International Journal of Theoretical Physics, Vol. 26, No. 4, 1987

Short-Time Evolution of Nonlinear Klein–Gordon Systems

C. Guha-Roy,[1] B. Bagchi,[1] and D. K. Sinha[1]

Received December 5, 1986

The short-time evolution of a class of nonlinear Klein–Gordon systems is studied. For nonzero mass, the short-time behavior of the field variable has an inverse-sine spectrum rather than an exponential one.

The short-time behavior of a certain class of nonlinear potentials has recently been studied by a number of authors (Bocchieri *et al.*, 1970; Casati *et al.*, 1980; Callegari *et al.*, 1979; Benettin *et al.*, 1980; Livi *et al.*, 1983, 1985a, b; Sulem *et al.*, 1983). Apart from deriving simple scaling laws, these studies have revealed that the analytic continuation of the field variable to the complex domain introduces singularities into the structure of the solutions. As such, a perturbative approach appears to be ruled out.

Most such studies on nonlinear systems have been carried out using standard computational techniques (Bocchieri *et al.*, 1970, Casati *et al.*, 1980; Callegari *et al.*, 1979; Sulem *et al.*, 1983). These systems include the Fermi–Pasta–Ulam (FPU) model (Fermi *et al.*, 1965) as well as those described by Lennard-Jones (Galgani and Lo Vecchio, 1979; Benettin *et al.*, 1980) and $\lambda\Phi^n$ potentials (Butera *et al.*, 1980). However, it has been pointed out (Livi *et al.*, 1983) that if one neglects the effects of all the linear terms, a class of nonlinear potentials may yield simple analytical solutions valid at very short time.

The purpose of this work is to study the short-time behavior of systems described by FPU and power law potentials with a view to obtaining more exact solutions. In this regard, we have considered the effects of the linear term involving the mass parameter and have obtained our solutions in closed form. An interesting characteristic of nonlinear Klein–Gordon as well as

[1]Department of Mathematics, Jadavpur University, Calcutta 700 032, Inda.

0020-7748/87/0400-0395$05.00/0

FPU potentials is that an analytical continuation of the field variables to the complex domain yields singularities that are just simple poles.

Consider a class of nonlinear Klein–Gordon potentials described by the Hamiltonian density

$$H(\pi, \Phi) = \frac{1}{2}\left[\pi^2 + \left(\frac{\partial \Phi}{\partial x}\right)^2 + m^2\Phi^2\right] + \frac{\lambda}{2n}\Phi^{2n} \tag{1}$$

where π represents the conjugate momentum of the field variable Φ and $n = 2, 3, \ldots$ stands for the possible degree of the potential. We assume that our field variable $\Phi(x, t)$ is defined on a finite-length interval $(0 \leq x \leq l)$ and is governed by the initial conditions of the type

$$\begin{aligned} \Phi(x, 0) &= A\cos(kx) \\ \dot{\Phi}(x, 0) &= 0 \end{aligned} \tag{2}$$

In (2), $k = 2\pi n/l$ and the dot denotes the partial differentiation with respect to time.

To obtain the short-time evolution of the solutions that follow from (1), the standard procedure (Sulem *et al.*, 1983) is to analytically continue $\Phi(x, t)$ to the complex function $\Psi(Z, t)$, where $Z = x + iy$. If now $\Psi(Z, t)$ is approximated by a form $1/(Z - Z_0)^\mu$, where Z_0 is a dominant singularity and μ is a constant, the Fourier transform of $\Phi(x, t)$ may be obtained as (Sulem *et al.*, 1983)

$$|\tilde{\Phi}(K, t)|^2 \sim K^{2(\mu-1)} \exp[-2K|y(t)|] \tag{3}$$

Clearly, the rhs of (3) represents an exponentially shaped spectrum. However, in the models that we explore below, we shall see that the short-time behavior of the field variable is governed by an inverse-sine curve (rather than an exponential spectrum) even in the presence of simple poles.

To see that (1) yields simple poles, we consider the equation of motion and analytically continue it to the complex plane.

Let us consider $n = 2$. From (1), the equation of motion may be obtained as

$$\Box\Phi = -m^2\Phi - \lambda\Phi^3 \tag{4a}$$

whose analytic continuation to the Z plane is

$$\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial Z^2} - m^2\Psi - \lambda\Psi^3 \tag{4b}$$

If the dominant singularity is of the form $1/(Z - Z_0)^\mu$, then it can be checked from (4), by comparing the leading terms, that $\mu = 1$. Thus, the singularity of (4) is indeed a simple pole.

Livi *et al.* (1985b) showed, by neglecting the effects of all the linear terms, that in the presence of singularities like simple poles, the time evolution of the system (4a) behaves like an exponential curve. A similar behavior has been found to hold for the FPU and Φ^6 theories. In the following, we retain the mass term in the equations and show by an explicit analytic evaluation that the exponential shape of the field variable at small times may get distorted for moderately large values of m^2.

An analytical continuation of (4) to the complex plane results in the following pair of coupled equations (by separating real and imaginary parts)

$$\ddot{\Psi}_R = \Psi_R'' - m^2\Psi_R - \lambda(\Psi_R^3 - 3\Psi_R\Psi_I^2) \tag{5a}$$

$$\ddot{\Psi}_I = \Psi_I'' - m^2\Psi_I + \lambda(\Psi_I^3 - 3\Psi_R^2\Psi_I) \tag{5b}$$

where a prime stands for the partial differentiation with respect to Z.

To enquire into the dynamics of the model, we set $\Psi_R = 0$ and note that at sufficently small times, the spatial derivative term Ψ_I'' will not contribute significantly (Livi *et al.*, 1985b). As a result, we obtain from (5)

$$\ddot{\Psi}_I = -m^2\Psi_I + \lambda\Psi_I^3 \tag{6}$$

Using conservation of energy, we can invert the above equation to give

$$t = \int_{\Psi_I(0)}^{\infty} \frac{d\Psi_I}{(-m^2\Psi_I^2 + \frac{1}{2}\lambda\Psi_I^4)^{1/2}} \tag{7}$$

where t stands for the time needed to reach infinity from the point $\Psi_R = 0$, $\Psi_I = \Psi_I(0)$. Interestingly, even in the presence of the m^2 term, the integral in (7) may be evaluated in a closed form to obtain

$$t = \frac{1}{m}\left\{\frac{\pi}{2} - \sec^{-1}\left[\frac{1}{m}\left(\frac{\lambda}{2}\right)^{1/2}\Psi_I(0)\right]\right\} \tag{8a}$$

which yields

$$\Psi_I(0) = m\left(\frac{2}{\lambda}\right)^{1/2}\frac{1}{\sin(mt)} \tag{8b}$$

To check the consistency of the solution (8) with the result obtained by Livi *et al.* (1985b), we apply the l'Hôspital rule to the rhs of (8a) and pass to the limit as $m \to 0$. It is readily seen that $\Psi_I(0)$ turns out to be $1/(\lambda^{1/2}t)$.

The initial conditions (2) allow one to have an estimate of $\operatorname{Im}\Psi(Z, 0)$ when the values of y are large. By analytically continuing (2) to the complex plane, $\operatorname{Im}\Psi(Z, 0)$ may be obtained as

$$\operatorname{Im}\Psi(Z, 0) \sim A\exp(ky) \tag{9}$$

for large y. We now postulate that the location of the singularity of $\Psi(Z, 0)$ will be at the same place where Im $\Psi(Z, 0)$ is roughly of the same order of magnitude as $\Psi_{\mathrm{I}}(0)$. We then obtain from (8b) and (9)

$$t=\frac{1}{m}\sin^{-1}\left[\frac{\sqrt{2}m}{A\sqrt{\lambda}}\exp(-ky)\right] \tag{10}$$

Thus, the shape of t is an inverse-sine curve and is seen to depend explicitly on the mass parameter m. It is to be stressed that m^2 in (1) need not be treated as a small parameter and as such may contribute significantly as in (10) even when $t\to 0$. Thus our result for t in (10) is to be considered as an improvement over what was obtained without the mass term (Livi *et al.*, 1985b), namely

$$t=\frac{1}{A\sqrt{\lambda}}\exp(-ky)$$

An analogous treatment to extract the short-time evolution of $\Phi(x, t)$ can also be applied to Klein-Gordon potentials corresponding to $n=3, 4, \ldots$.

We now consider the FPU model, whose Hamiltonian density is given by

$$H=\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial t}\right)^2+\left(\frac{\partial\Phi}{\partial x}\right)^2\right]+\frac{\beta}{4}\left(\frac{\partial\Phi}{\partial x}\right)^4 \tag{11}$$

where β is a parameter.

It is not difficult to see that the equation of motion corresponding to (11) may be reduced (Livi *et al.*, 1983, 1985b), by using (2), into the form

$$\frac{\partial^2}{\partial t^2}\eta(x, t)=-C_1\eta(x, t)-C_2\eta^3(x, t) \tag{12}$$

for sufficiently small times. In (12), $\eta(x, t)=(\partial/\partial x)\Phi(x, t)$ and the parameters C_1 and C_2 stand for

$$C_1=k^2(1-6\beta A^2k^2), \qquad C_2=9\beta k^2 \tag{13}$$

As before, we now split η into a real part η_{R} and an imaginary part η_{I}. The analog of (6) then reads

$$\ddot{\eta}_{\mathrm{I}}=-k^2(1-6\beta A^2k^2)\eta_{\mathrm{I}}+9\beta k^2\eta_{\mathrm{I}}^3 \tag{14}$$

which may be readily solved to obtain

$$\eta_{\mathrm{I}}(0)=\frac{1}{3}\left[\frac{2}{\beta}(1-6\beta A^2k^2)\right]^{1/2}\frac{1}{\sin[k^2(1-6\beta A^2k^2)]^{1/2}t} \tag{15}$$

By combining the above relation for $\eta_I(0)$ with the imaginary part of the analytic continuation of the initial condition (2) in the complex domain, we find, for large y, that in the FPU case, too, the shape of t turns out to be an inverse-sine curve, namely

$$t = \frac{1}{k(1-6\beta A^2k^2)^{1/2}} \sin^{-1}\left[\frac{\sqrt{2}}{3Ak\sqrt{\beta}}(1-6\beta A^2k^2)^{1/2}\exp(-ky)\right] \quad (16)$$

As in the $\lambda\Phi^4$ case, here also the shape of t is sensitive to the coefficient of the η_I term in (14).

To summarize, we have shown that it is possible to solve analytically the equations of motion of a class of nonlinear wave equations even in the presence of the mass term, which is essentially linear in character. Our studies of $\lambda\Phi^n$ and FPU types of potentials have shown that the short-time behavior of the field variable obeys an inverse-sine curve rather than an exponentially shaped spectrum as obtained in Livi *et al.* (1983, 1985b). The reason lies in the fact that in all these studies, the linear part of the equation of motion involving the mass parameter has been neglected. Since there is no justification for deleting such terms even for sufficiently short times, the exact results reported here should provide a better understanding of the dynamics of nonlinear systems.

ACKNOWLEDGMENTS

This work was supported by the Council of Scientific and Industrial Research and the University Grants Commission (Department of Special Assistance Programme), New Delhi.

REFERENCES

Benettin, G., Lo Vecchio, G., and Tenenbaum, A. (1980). *Physical Review A*, **22**, 1709.

Bocchieri, P., Scotti, A., Bearzi, B., and Loinger, A. (1970). *Physical Review A*, **2**, 2013.

Butera, P., Galgani, L., Giorgilli, A., Tagliani, A., and Sabata, H. (1980). *Nuovo Cimento*, **59B**, 81.

Callegari, B., Carotta, M. C., Ferrario, C., Lo Vecchio, G., and Galgani, L. (1979). *Nuovo Cimento*, **54B**, 463.

Casati, G., Chirikov, B. V., and Ford, J. (1980). *Physics Letters*, **77A**, 91.

Fermi, E., Pasta, J., and Ulam, S. (1965). In *Collected Papers*, E. Fermi ed., p. 978, University of Chicago Press, Chicago.

Galgani, L., and Lo Vecchio, G. (1979). *Nuovo Cimento*, **52B**, 1.

Livi, R., Pettini, M., Ruffo, S., Sparpaglione, M., and Vulipani, A. (1983). *Physical Review A*, **28**, 3544.

Livi, R., Pettini, M., Ruffo, S., Sparpaglione, M., and Vulpani, A. (1985a). *Physical Review A*, **31**, 1039.

Livi, R., Ruffo, S., Pettini, M., and Vulpiani, A. (1985b). *Nuovo Cimento*, **89B**, 120.

Sulem, C., Sulem, P. M., and Frisch, H. (1983). *Journal of Computational Physics*, **50**, 138.